Confronting Elusive Insurgents

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April 7, 2008

Abstract

During counterinsurgency operations, government forces with superior firepower confront weaker low-signature insurgents that exercise elusive yet effective "strike-and-hide" tactics. Under what conditions should government forces attack insurgent strongholds? How should the government allocate its force across different strongholds when the insurgents' threat to the population must be taken into account? How should the government respond to "smart" insurgents who anticipate the government's optimal plan of attack and prepare accordingly? This article addresses these questions. Using Lanchester models modified to account for im-

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Report Documentation Page

1. REPORT DATE	2. REPORT TYPE	3. DATES COVERED
7 APR 2008	**N/A**	**-**

4. TITLE AND SUBTITLE	5a. CONTRACT NUMBER
Confronting Elusive Insurgents	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)	8. PERFORMING ORGANIZATION REPORT NUMBER
Naval Postgraduate School Operations Research Department Monterey, CA 93943	

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release, distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT
see report

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE	**SAR**	**46**	
unclassified	**unclassified**	**unclassified**			

perfect intelligence, we formulate an optimal force allocation problem for the government and develop a knapsack approximation that has tight error bounds. We also model a sequential force allocation game between the insurgents and the government and solve for its equilibrium. Under reasonable assumptions regarding government behavior, it is optimal for the insurgents to "spread out" in a way that maximizes the number of soldiers required to win all battles. When the government follows a knapsack strategy, the insurgents induce the government to select battles in the worst possible order. Such strategic insurgent behavior reduces the extent to which the government can prevent civilian and battle casualties.

1   Introduction and Motivation

Many recent military conflicts can be characterized as asymmetric combat situations pitting government forces against guerilla, insurgent or terrorist organizations. In Afghanistan, US forces have been fighting the Taliban and other fundamentalist groups for several years [Barno 2007]; in Iraq, coalition forces are confronting both Sunni and Shiite insurgencies [Hoffman 2004]-[Cordesman and Davies 2007]; and in Colombia a mixture of armed leftist guerrilla groups and drug barons challenge the local government with continuous violence [Rabasa and Chalk 2001]. In the summer of 2006, following the kidnapping of two Israeli soldiers and missile attacks on northern Israel, the Israel Defense Forces (IDF) launched an attack on the Hezbollah in Southern Lebanon. Although the force ratio was in favor of the Israelis by almost an order of magnitude, the outcome of this war

was at best mixed from the Israeli perspective. There was no decisive victory [Cordesman 2007], [Blanford 2006]. The Hezbollah combatants, diffused and hidden in defensive positions in several villages and small towns, were elusive targets for the superior Israeli fire power. The IDF troops had the military means and capabilities to effectively engage the Hezbollah targets, but simply could not find most of them. While inflicting heavy casualties on Hezbollah, the IDF forces were unsuccessful in capturing the villages, and this failure to eliminate the guerrilla units resulted in a continuous barrage of missiles on northern Israel. This example reflects the key advantage of insurgent forces, namely, their elusiveness and low target signature which facilitate effective "strike-and-hide" tactics.

To model specific battles between government troops and insurgents, we employ Lanchester-style models [Lanchester 1916] where individual battles are described by sets of differential equations. Lanchester models have been used in the past to describe guerrilla warfare. Deitchman [Deitchman 1962] presented the first such model, which was followed by Schaffer [Schaffer 1968]. The engagement dynamics in these papers is asymmetric and involve a mixture of precise (*aimed*) and imprecise (*area)* fire: guerrillas can observe the movement and location of government forces and engage with aimed fire, while government forces have limited situational awareness and therefore engage with area fire. Aimed fire leads to the Lanchester Square Law, while area fire leads to the Lanchester Linear Law [Morse and Kimball 1946].

There are very few models addressing the role of intelligence in general com-

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bat, and none in the context of guerrilla or counterinsurgency warfare. A first attempt to incorporate the effect of situational awareness, command and control and intelligence in a Lanchester model is reported in [Schreiber 1964], where these capabilities refer to the ability to distinguish between surviving and killed targets. The effect of intelligence on the distribution and specific types of air-to-air, ground-to-air, and air-to ground engagements is discussed in [Allen 1993].

The Lanchester models in this paper extend Deitchman's approach in two ways. First, we incorporate an explicit intelligence function that manifests the targeting capabilities of government forces against the insurgents. Second, we employ a damage function that accounts for civilian casualties inflicted by the insurgents in addition to government force battle casualties.

In this paper, we model government attacks on insurgent strongholds. We use the term *stronghold* for a village or town in which armed insurgents are present. We develop models for optimally allocating government forces to attack insurgents dispersed in several strongholds to minimize the total number of casualties (civilian and soldiers) caused by the insurgents. We also use game theory to model "smart" insurgents who are able to anticipate optimal government countertactics and deploy in a manner that leads to worst-case results for the government.

We show that under reasonable assumptions regarding government behavior, it is optimal for insurgencies to "spread out" and induce government attacks on multiple strongholds to maximize the damage the insurgents can inflict upon the government. Not surprisingly, improved intelligence and greater total force

are the main levers by which the government can reduce total casualties.

The paper proceeds as follows: in the next section we develop the intelligence-dependent Lanchester model and define the associated damage function. In Section 3 we consider the problem of force allocation when the government knows how many insurgents are present in each stronghold, but does not know insurgents' locations with certainty within strongholds. We develop an easy-to-implement knapsack allocation policy for the government, illustrate this model in a symmetric case that yields closed-form solutions for insight, and consider a more realistic "Towns and Villages" example based on the 2006 Lebanon war. In Section 4, we study a sequential game that ensues when the insurgents anticipate government actions and plan accordingly. We show that if the government follows the knapsack policy of Section 3, the insurgents can prepare by allocating their fighters across strongholds to maximize the number of soldiers required to win all battles, and inducing the government to select a poor knapsack sequence. We illustrate this game by building on our earlier example. Section 5 presents concluding remarks, while mathematical proofs are given in the Appendix.

2   Lanchester Battles and Damage Function

Consider a government force of size x_0 that engages a stronghold held by y_0 insurgents. The insurgents have perfect information regarding the location and movement of the government force and therefore engage with aimed fire. The effectiveness of the government force, however, is limited by the intelligence

governing the precise locations of insurgents within the stronghold. Let $x(t)$ and $y(t)$ denote the surviving number of government soldiers and the insurgents, respectively, at time t. We model this situation using Lanchester-style equations modified to account for the asymmetric nature of counterinsurgency operations. Specifically,

$$\frac{dx(t)}{dt} = -\beta y(t) \tag{1}$$

and

$$\frac{dy(t)}{dt} = -\alpha x(t)p(y(t)) \tag{2}$$

where α and β are the attrition rates of insurgents and soldiers respectively. The intelligence function $p(y)$ models the per-shot probability of targeting an insurgent. This function depends on the degree of intelligence the soldiers possess regarding insurgent deployment as well as the number of surviving insurgents at the time of fire. We will discuss a specific form of $p(y)$ later on, but first we present more general results.

We presume for simplicity that the battle ends when one side eliminates all those on the other.[1] The analysis proceeds in standard fashion by noting that

$$\delta(t) = \frac{\alpha}{2}x^2(t) - \beta \int_0^{y(t)} \frac{u}{p(u)} du \tag{3}$$

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[1] The model can be modified to account for limited endurance on both sides, meaning that the combatants are only willing to endure a certain number of casualties before abandoning the fight.

equals a constant with respect to time, say δ, for from (1)-(2),

$$\frac{d\delta(t)}{dt} = \alpha x(t)\frac{dx(t)}{dt} - \beta\frac{y(t)}{p(y(t))}\frac{dy(t)}{dt} = 0. \tag{4}$$

Equation (4) must also hold at time 0, which implies that

$$\delta(0) \equiv \delta = \frac{\alpha}{2}x_0^2 - \beta\int_0^{y_0}\frac{u}{p(u)}du. \tag{5}$$

The government soldiers are victorious if the insurgents are vanquished leaving a positive number of surviving government troops. This can only occur if $\delta > 0$, and from equation (5), this criterion for government victory over the insurgents is equivalent to

$$x_0 > \sqrt{\frac{2\beta}{\alpha}\int_0^{y_0}\frac{u}{p(u)}du} \equiv B. \tag{6}$$

where B is the *victory threshold* – the minimum number of soldiers needed to defeat the y_0 insurgents. Via equations (3)-(5), when the government wins the surviving number of soldiers x_s satisfies

$$\frac{\alpha}{2}x_s^2 - 0 = \delta = \frac{\alpha}{2}(x_0^2 - B^2) \tag{7}$$

which yields

$$x_s = \sqrt{x_0^2 - B^2} \tag{8}$$

while $x_0 - x_s$ soldiers fall in the battle.

Suppose that if the soldiers are not victorious, the insurgents cause k civilian casualties. For example, as in the Second Lebanon War between Hezbollah and Israel, the stronghold could be the source of missiles fired upon the general population that would cause damage equivalent to losing k civilian lives. Or, the

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stronghold could be a base from which insurgents initiate improvised explosive device (IED), suicide bombing or other attacks targeting civilians. Whether it is worthwhile to attack the insurgents no longer depends solely upon whether or not victory can be achieved but also on the total number of casualties – civilians and soldiers. With equivalent valuation of civilians and soldiers,[2] and utilizing (8), the damage function representing the total number of casualties as a function of the size of the government force x_0 is given by

$$d(x_0) = \begin{cases} k + x_0 & x_0 \leq B \\ \\ x_0 - \sqrt{x_0^2 - B^2} & x_0 > B \end{cases} . \tag{9}$$

A plot of $d(x_0)$ with $B = 100$ and $k = 50, 100$ and 150 appears in Figure 1.

Figure 1 Here

To ensure that an attack on the insurgent stronghold saves more downstream casualties than are lost in the attack, that is, $d(x_0) < k$, the size of the attacking force must be sufficiently large. This condition is satisfied when

$$x_0 > \begin{cases} \frac{k}{2} + \frac{B^2}{2k} & k \leq B \\ \\ B & k > B \end{cases} . \tag{10}$$

Inequality (10) shows that the minimum size of the attacking force required to ensure that $d(x_0) < k$ is a *decreasing* function of k for $k \leq B$, and constant

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[2] A simple extension of the argument to follow applies if civilian or other downstream casualties are valued differently relative to casualties among the attacking government force.

thereafter. This argues against attacking insurgents where the benefits of success are slim (k is small) even if the ability to succeed is apparent (at least B soldiers can be deployed in the attack), for absent overwhelming force, the number of soldiers lost in combat could exceed the downstream casualties averted by defeating the insurgents. Note that in Fig. 1, the minimum troops to be allocated equals 100 when $k = 100$ and 150, but 125 when $k = 50$ in accord with inequality (10).

3 Optimal Force Allocation

Suppose that the insurgents are dispersed in m strongholds, where the ith stronghold is defended by y_i insurgents, and conquering it would avert k_i civilian casualties. The government has two decisions to make: which (if any) strongholds to attack, and how many soldiers x_i to allocate to each stronghold i pursued. In addition to the force sizes x_i and y_i, the battle conditions in the ith stronghold are determined by the attrition rates α_i and β_i and by the intelligence function $p_i(y)$. Recall that the latter four parameters determine the victory threshold B_i (equation (6)). The government has a total force of size f to allocate to battles in the various strongholds. To keep the notation simple, without loss of generality, we assume for the remainder of this article that

$$\frac{k_1}{B_1} \geq \frac{k_2}{B_2} \geq \cdots \geq \frac{k_m}{B_m}. \tag{11}$$

As a step towards determining the optimal force allocation, we initially assume that the set of strongholds to be engaged, V, is given, and that f is

sufficiently large to defeat all insurgent strongholds in that set (will relax these assumptions shortly). Since all strongholds in V are defeated, there are no civilian casualties emanating from those strongholds, thus minimizing the total number of casualties is equivalent to maximizing the number of surviving soldiers. The latter is determined from equation (8) for each battle, which leads to the following optimization problem:

$$\max \sum_{i \in V} \sqrt{x_i^2 - B_i^2} \tag{12}$$

$$\text{st} \quad \sum_{i \in V} x_i = f. \tag{13}$$

The solution to this problem, which is easily found by placing a Lagrange multiplier on the total force constraint and differentiating, is given by

$$x_i^* = \frac{B_i}{\sum_{j \in V} B_j} f \quad \text{for } i \in V. \tag{14}$$

Equation (14) shows that given a set of strongholds to engage, the optimal allocation of soldiers is proportional to B_i; the higher the victory threshold the greater the number of soldiers allocated to that battle. In particular, the higher the quality of tactical intelligence in a certain stronghold, the fewer soldiers are allocated. We have thus reduced the problem to determining the set V. To do so, let the binary variables $V_i = 1$ if stronghold i is attacked, $V_i = 0$ otherwise,

and consider the following optimization problem:

$$\min \sum_{i=1}^{m} k_i(1 - V_i) + \sum_{i=1}^{m} \left(x_i - \sqrt{x_i^2 - B_i^2} \right) V_i \qquad (15)$$

$$\text{st} \quad x_i \sum_{j=1}^{m} B_j V_j = B_i f \ V_i \qquad \text{for } i = 1, 2, ..., m \qquad (16)$$

$$V_i \in (0, 1) \qquad \text{for } i = 1, 2, ..., m \qquad (17)$$

The objective function (15) is the total damage (see (9)) across all battles. Constraint (16) ensures that if the ith stronghold is attacked, then the number of soldiers allocated to that battle will follow (14), while if the ith stronghold is not attacked, then no soldiers are allocated there.

3.1   Knapsack Approximation to Optimal Force Allocation

While the formulation (15-17) above can be employed to determine optimal force allocations in any particular instance, we can develop greater insight with an approximation that yields analytical results. In the optimal solution, soldiers are allocated to battles within the optimal set V in accordance with equation (14); here we develop a fast method for approximating the optimal battle set.

Let

$$K = \frac{f}{\sum_{j \in V} B_j} \qquad (18)$$

and write

$$x_i^* = \begin{cases} KB_i & i \in V \\ \\ 0 & i \notin V \end{cases} . \qquad (19)$$

Consider the following integer knapsack conditional on K:

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$$\max \sum_{i=1}^{m} \left\{ k_i - KB_i + \sqrt{(KB_i)^2 - B_i^2} \right\} V_i \qquad (20)$$

$$\text{st} \quad K \sum_{i=1}^{m} B_i V_i = f \qquad (21)$$

$$V_i \in (0, 1) \qquad i = 1, 2, ..., m \qquad (22)$$

The objective function (20) is derived from (15), (18) and (19), and it represents the total casualties *averted* by attacking the insurgents. Constraint (21) defines the relation between K and the V_i variables.

Rather than solving (20-20) exactly, we will solve its continuous relaxation. Consider the ratio

$$r_i = \frac{k_i - KB_i + \sqrt{(KB_i)^2 - B_i^2}}{KB_i} = \frac{k_i}{KB_i} - 1 + \frac{\sqrt{K^2 - 1}}{K} \qquad (23)$$

and note that $r_i > r_j$ if and only if $k_i/B_i > k_j/B_j$. For a given K, The continuous relaxation to problem (20)–(22) is found simply by rank ordering the targets from largest to smallest r_i, and choosing the first $j^*(K)$ battles in this ordering to fight, where $j^*(K)$ is the largest value of j such that $\sum_{i=1}^{j^*(K)} KB_i \leq f$ within the ranking. The optimal force allocations for this conditional (on K) knapsack are simply

$$x_i^*(K) = \begin{cases} KB_i & i \leq j^*(K) \\ \\ 0 & i > j^*(K) \end{cases} \qquad (24)$$

To find the optimal value of K, first note that we only need to consider values of K that exhaust the force with equality (since the entire force will be

expended in the optimal solution). Let K_j correspond to that value of K that would allocate soldiers to the first j battles within the knapsack ranking. From the force allocation constraint we have

$$K_j = \frac{f}{\sum_{h=1}^{j} B_h} \quad \text{for } j = 0, 1, 2, ..., m. \tag{25}$$

When $K = K_j$, equation (24) becomes

$$x_i^*(K_j) = \begin{cases} \frac{B_i}{\sum_{h=1}^{j} B_h} f & i \leq j \\ \\ 0 & i > j \end{cases} \tag{26}$$

Now, substitute equation (26) into equation (20) after replacing m by j, K by K_j (using equation (25) and setting all the V_i's $= 1$ to obtain

$$\max_{0 \leq j \leq j^+} \sum_{i=1}^{j} k_i - f + \sqrt{f^2 - (\sum_{i=1}^{j} B_i)^2}; \tag{27}$$

where j^+ is the largest value of j such that $\sum_{i=1}^{j} B_i \leq f$, and $j = 0$ corresponds to the decision not to fight any battles (in which case no casualties are averted). This problem is easily solved by enumeration over j; given the value j^* that maximizes (27), the knapsack-optimal allocation is given by

$$x_i^* = \begin{cases} \frac{B_i}{\sum_{h=1}^{j^*} B_h} f & i \leq j^* \\ \\ 0 & i > j^* \end{cases} \tag{28}$$

In the appendix we present conditions under which the approximation error from optimizing (27) instead of (15) becomes negligibly small.

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3.2 Example 1: Symmetric Insurgents $(k_i = k,\ B_i = B)$

Consider the special case where $k_i = k$ and $B_i = B$ for all battles. Then the objective in (27) becomes

$$\max_j jk - f + \sqrt{f^2 - j^2 B^2}. \tag{29}$$

Treating (29) as a continuous optimization problem, we obtain that the optimal number of strongholds to attack is

$$j^* = \sqrt{\frac{k^2}{B^2 + k^2}} \frac{f}{B} \tag{30}$$

which is assured to be smaller than the maximum number of battles that could be fought, $j^+ = f/B$. The optimal number of soldiers allocated per battle fought equals

$$x^* = \frac{f}{j^*} = \sqrt{\frac{B^2 + k^2}{k^2}} B. \tag{31}$$

Total casualties averted are found by substituting j^* into the objective function (29) and equal

$$\text{Casualties Averted} = \left(\sqrt{\frac{B^2 + k^2}{B^2}} - 1\right) f. \tag{32}$$

These results accord with intuition: the optimal number of battles attenuates the maximum number of battles that could be fought (f/B) by the factor $\sqrt{\frac{k^2}{B^2 + k^2}}$. If k is very large relative to B, this factor tends towards one and it is optimal to fight as many battles as possible to avert large number of civilian casualties. If k is very small relative to B, then it is not optimal to fight *any* battles $(j^* \to 0)$ as battle casualties would exceed the downstream casualties averted.

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3.3 Example 2: Insurgents in Towns and Villages

Consider an attack on a town where y insurgents grouped in cells of c insurgents per cell are dispersed among ℓ distinct sites (houses, buildings) within the town. There are y/c different cells spread amongst the ℓ sites, thus the probability that a randomly selected structure contains insurgents is equal to $(y/c)/\ell = y/n$ where $n = c\ell$.

This example is based on general and partial data regarding the 2006 war between Israel and Hezbollah in Southern Lebanon. A typical village in southern Lebanon (e.g. Maroun-A Ras, Yaroun) contains $100 - 200$ houses and buildings, while within a small town (e.g., Bint Jbail, Tyre), $800 - 1,500$ structures are typical [Google Earth 2008]. We consider attacking up to five insurgent strongholds (three villages and two towns) where initially the insurgents operate in cells of size $c = 10$. Left to their own devices, insurgents in the ith stronghold are capable of inflicting k_i civilian casualties. The data regarding these five strongholds are summarized in Table 1.

Table 1 Here

Next we specify the intelligence function $p(y)$. Absent intelligence, the soldiers fire at random (area fire), thus only a fraction y/n of their fire is effective. In this (worst) case, $p(y) = y/n$, and from (2) we obtain the Deitchman model with

$$B = \sqrt{\frac{2\beta n y}{\alpha}}. \tag{33}$$

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[Deitchman 1962]. At the other extreme, if the soldiers have perfect intelligence then $p(y) = 1$ and one obtains the Lanchester direct-fire model with

$$B = \sqrt{\frac{\beta}{\alpha}} y. \tag{34}$$

Since it is always possible to fire at random, we assume that $y/n \leq p(y) \leq 1$. As an intermediate formulation, we use a linear function [Kress and Szechtman 2008]

$$p(y) = \mu + (1 - \mu)\frac{y}{n}. \tag{35}$$

This function has an intuitive interpretation: the soldiers know the location of a fraction μ of the insurgents and therefore engage them by direct fire, while the remaining fraction $(1 - \mu)$, whose locations are not known, are engaged by area fire. Note that $\mu = 0$ replicates the Deichtman model while $\mu = 1$ replicates the original Lanchester model. Via equation (6), the threshold value B for this linear intelligence function is given by

$$B = \sqrt{\frac{2\beta n}{\alpha(1 - \mu)} \left[y - \frac{n\mu}{(1 - \mu)} \log\left(\frac{(1 - \mu)y}{\mu n} + 1 \right) \right]}. \tag{36}$$

We assume that the effectiveness ratio β/α equals 0.5 for all five strongholds attacked, which means that the (nominal) attrition rate of the insurgents (by the soldiers) is twice the attrition rate of the soldiers. Table 2 reports the maximal casualties averted and optimal soldiers allocations to the five different battles as a function of the intelligence parameter μ when the total government force is constrained to $f = 2,000$ (about two brigades). Note that for any value of μ, ranking the strongholds from largest to smallest ratio of k/B yields the

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ordering: Town 1, Town 2, Village 2, Village 1 and Village 3. In this example, the knapsack approximation of equations (27)-(28) yields the correct optimal results. When $\mu = 0$ (no tactical intelligence) the optimal strategy is attack Town 1 with full force while forfeiting Town 2 and the villages to the insurgents; this serves to prevent only 12 soldiers and civilian casualties in total, but is the best result possible. As μ increases, and tactical intelligence improves, the government is able to engage more of the insurgent strongholds; indeed once μ reaches 0.5, all strongholds are attacked. The results also indicate an astonishing initial return in casualties averted to intelligence – simply knowing where 5% of the insurgents enables a tenfold increase in the number of casualties prevented compared to no intelligence (from 12 to 122). Diminishing returns set in thereafter, showing that while minimal intelligence is greatly preferred to none, perfect intelligence only leads to a 10% improvement in casualties averted compared to "reasonable" intelligence (e.g., $\mu = 0.65$).

Table 2 Here

Figure 2 reports the maximal number of casualties averted and optimal force allocations across the five insurgent strongholds. We vary the overall size of the attacking force f from 0 to 2,000 while holding all other parameters constant at their earlier values and fixing $\mu = 0.5$. Figure 2 shows the scalloping, piecewise-concave nature of the optimal casualties averted function while reporting the split of the total force in proportion to those B_i's in the optimal battle set V. The sharpest growth in casualties averted occurs when $199 \leq f \leq 546$ and only Town 1 is attacked. At small total force levels ($73 \leq f \leq 145$) only Village 3 is

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attacked (for $f \leq 72$, *no* strongholds are attacked as launching any strikes would increase total casualties beyond $\sum_i k_i = 520$), while for $146 \leq f \leq 198$ only Village 2 is attacked. The knapsack approximation produces optimal results for all values of $f \geq 670$, but for smaller values of f the true optimal decisions depart from the knapsack values. For example, the knapsack approximation does not recommend any attacks until $f = 200$, at which point Town 1 is attacked, while for $547 \leq f \leq 669$ when the optimal decision is to attack Town 1 and Village 2, the knapsack approximation recommends attacking only Town 1 or Towns 1 and 2, depending upon the value of f). The largest percentage error in the total number of casualties between the optimal and knapsack results equals 4.9% when $f = 198$ (the optimal strategy is to attack Village 2, but the knapsack recommends abstaining from any attacks, forgoing the opportunity to avert 24 casualties).

<div align="center">

Figure 2 Here

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4   Sequential Force Allocation Game

So far we have assumed that only the government faces the force allocation problem. However, if the insurgents knew about the government's situation, then they would allocate their resources (human and materiel) in order to inflict the largest possible amount of damage. While in reality the insurgents don't know all of the government's problem parameters (intelligence function, attrition coefficients, etc.), a zero-sum sequential game provides a worst-case scenario for

<div align="center">18</div>

the government.

Suppose the insurgents have a total of ψ fighters, which results in the force levels $\mathbf{y} = (y_1, \ldots, y_m)$ belonging to the simplex $\mathcal{M}_\psi = \{\mathbf{y} \geq \mathbf{0}; \sum_{i=1}^{m} y_i \leq \psi\}$. Also, suppose that the insurgents may inflict at most κ civilian casualties, meaning that $\mathbf{k} = (k_1, \ldots, k_m)$ belongs to the simplex $\mathcal{M}_\kappa = \{\mathbf{k} \geq \mathbf{0}; \sum_{i=1}^{m} k_i \leq \kappa\}$. Faced with an insurgency spread according to certain $\mathbf{y}$ and $\mathbf{k}$, the government allocates its forces following equation (14) and decides where to fight by solving problem (15–17), leading to $\eta(\mathbf{y}, \mathbf{k})$ casualties in total. Thus, the insurgents' problem is to

$$\max \eta(\mathbf{y}, \mathbf{k}) \tag{37}$$

$$\text{st} \quad \mathbf{y} \in \mathcal{M}_\psi \tag{38}$$

$$\mathbf{k} \in \mathcal{M}_\kappa \tag{39}$$

Clearly $\mathbf{y}$ and $\mathbf{k}$ are such that $\sum_{i=1}^{m} y_i = \psi$ and $\sum_{i=1}^{m} k_i = \kappa$, because the insurgent's objective is making things worse for the government. Also, $k_i > 0$ if and only if $y_i > 0$. Indeed, $k_i > 0$ and $y_i = 0$ cannot be optimal because the k_i's would be destroyed by the soldiers at zero cost, and $k_i = 0$ and $y_i > 0$ cannot be optimal because the government would never attack such a stronghold. Note that since the soldiers can always choose not to engage the insurgents, it follows that $\eta(\mathbf{y}, \mathbf{k}) \leq \kappa$ for any feasible allocation of $\mathbf{y}$ and $\mathbf{k}$. Moreover, if $f \geq \sum_{i=1}^{m} B_i(y_i)$ for any feasible $\mathbf{y}$ then $\eta(\mathbf{y}, \mathbf{k}) \leq \max_{\mathbf{y} \in \mathcal{M}_\psi} \sum_{i=1}^{m} B_i(y_i)$.

The standard approach to solving problem (37–39) is to introduce a constraint for each feasible configuration of engagement variables $V_1, \ldots, V_m$. These

constraints determine a feasible set that is convex, since it is easy to see that each constraint is convex. Unfortunately, there are 2^m non-linear constraints, which makes it impossible to obtain an analytical solution without imposing more structure on the problem.

With this in mind, we observe that a drawback of constraint (38) is that an allocation of insurgents $\mathbf{y}$ matters insofar as it impinges on the victory thresholds $B_1, \ldots, B_m$ (equation (6)). This suggests that a more natural way to express constraint (38) is via a constraint on the victory thresholds B. Letting Γ be the set mapped by $B_1(y_1), \ldots, B_m(y_m)$ for all $\mathbf{y} \in \mathcal{M}_\psi$, we have the following result. (All proofs appear in the Appendix).

Lemma 1 *If the threshold functions $B_1(y_1), \ldots, B_m(y_m)$ are concave, then the set Γ is convex.*

From Equation (6), it easily follows (by taking the second derivative of $B(y)$) that convex intelligence functions $p(y)$ lead to concave threshold functions $B(y)$; this includes the linear intelligence function $p(y) = \mu + (1 - \mu)y/n$ of Section 3.3. Indeed, all the monomials $p(y) = \text{constant} \times y^\theta$ with $\theta \geq 0$ produce $B(y)$ concave. Hence, for the rest of this section we shall restrict our attention to convex sets Γ.

Looking at problem (37–39), the simplest scenario occurs when $p_i(y) = 1$, that is, the government has perfect intelligence for all i's. In this case $B(y)$ is a linear function of y, implying that there exists an extreme point – a stronghold i^* – such that $B_{i^*}(\psi) = \max_{\mathbf{y} \in \mathcal{M}_\psi} \sum_i B_i(y_i)$. That is, the sum of the thresholds is largest when the insurgents concentrate all their forces ψ in one stronghold,

i^*. In this case, there are several possible situations, depending on the relative values of κ, $B_{i^*}(\psi)$, and f. The results for each case are summarized in the next lemma.

Lemma 2 *Suppose $p_i(y) = 1$ for all i's. Then it is optimal for the insurgents to allocate all their resources to y_{i^*} and k_{i^*}, and there exist three possibilities:*

(i) If $f \geq \kappa \geq B_{i^*}(\psi)$ or $\kappa \geq f \geq B_{i^*}(\psi)$, the government engages the insurgents by setting $x_{i^*} = f$.

(ii) If $f \geq B_{i^*}(\psi) \geq \kappa$, the government engages the insurgents by setting $x_{i^*} = f$ if and only if $f - (f^2 - B_{i^*}^2(\psi))^{1/2} \leq \kappa$.

(iii) If $B_{i^*}(\psi) > f$, the government never engages the insurgents.

The practical implication of having $p_i(y) = 1$ is that the insurgency is under direct government fire. In this situation, Lemma 2 asserts that the insurgents concentrate their forces and that, depending on the force level f relative to κ and $B_{i^*}(\psi)$, the government engages the insurgents (when battle casualties are lower than civilian casualties prevented) or not. This conclusion is consistent with the principle of force concentration, which is derived from Lanchester's Square Law [Morse and Kimball 1946].

4.1 Knapsack Approximation Game

In Section 3 we argued that problem (27) provides a very good approximation to the allocation problem (15–17) faced by the government. This motivates analyzing the sequential zero-sum game that emerges when the insurgency allocates

its resources knowing that the government employs the knapsack approximation (27) to deploy its forces. Looking at problem (27), the insurgency problem is to

$$\min \theta(\mathbf{y}, \mathbf{k}) \tag{40}$$

$$\text{st} \quad \mathbf{y} \in \mathcal{M}_\psi \tag{41}$$

$$\sum_{i=1}^{m} k_i = \kappa, \quad k_i \geq 0 \tag{42}$$

where $\theta(\mathbf{y}, \mathbf{k}) = \max_{0 \leq j \leq j^+} \sum_{i=1}^{j} k_i - f + (f^2 - (\sum_{i=1}^{j} B_i(y_i))^2)^{1/2}$. Let $j^*(\mathbf{y}, \mathbf{k})$ be the optimal value of j when viewed as a function of $(\mathbf{y}, \mathbf{k})$, and recall that our rank-ordering assumption, equation (11), is in force. Finally, let $(\tilde{\mathbf{y}}, \tilde{\mathbf{k}})$ be an optimal allocation for the insurgency, and write $\tilde{B}_i = B_i(\tilde{y}_i)$, $B_i = B_i(y_i)$, for $i = 1, \ldots, m$.

To breach into problem (40–42), notice that since the government may choose to bear the civilian casualties without fighting (i.e., $j = 0$), we have $\theta(\mathbf{y}, \mathbf{k}) \geq 0$. Thus, any allocation $(\mathbf{y}, \mathbf{k})$ that achieves this lower bound is optimal. Also, the best the government hope for is to avert κ casualties at low soldiers' casualties, so that $\theta(\mathbf{y}, \mathbf{k}) \leq \kappa$.

Recall that the government selects the towns to engage or not in decreasing order of their ratios k_i/B_i. This suggests that a judicious approach for the insurgency is to *flatten* the ratios as much as possible. As the next lemma demonstrates, a necessary condition for insurgent-optimality when the number of potential civilian casualties κ is large, is to equalize these ratios. For κ sufficiently low (i.e., the government does not attack and $j^* = 0$), we show that there exist optimal solutions with k_i/B_i constant.

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Lemma 3 *Suppose the government allocates its forces by solving problem (27).*
If $\theta(\tilde{\mathbf{y}}, \tilde{\mathbf{k}}) > 0$, then the ratios $\tilde{k}_i / \tilde{B}_i$ must be constant, with

$$\tilde{k}_i = \kappa \frac{\tilde{B}_i}{\sum_{j=1}^{m} \tilde{B}_j}.$$

If $\theta(\tilde{\mathbf{y}}, \tilde{\mathbf{k}}) = 0$, then there exists a solution $(\mathbf{y}', \mathbf{k}')$ with constant ratios $k_i' / B_i'(y_i)$
and $\theta(\mathbf{y}', \mathbf{k}') = 0$.

Hence, since we can restrict attention to solutions $(\mathbf{y}, \mathbf{k})$ with constant ratios k_i / B_i, problem (40–42) becomes

$$\min_{0 \leq j \leq j^+} \max \quad \kappa \frac{\sum_{i=1}^{j} B_i}{\sum_{i=1}^{m} B_i} - f + \left(f^2 - \left(\sum_{i=1}^{j} B_i \right)^2 \right)^{1/2} \tag{43}$$

$$\text{st} \quad \sum_{i=1}^{m} y_i = \psi, \ y_i \geq 0. \tag{44}$$

It is easier to tackle (43 – 44) by considering the government's problem conditioned on $\sum_{i=1}^{m} B_i \equiv \ell$. Given ℓ, let j_ℓ be the optimal number of strongholds engaged by the government, and let $\phi_\ell(\sum_{i=1}^{j} B_i)$ be the right hand side of (43) (minus the "$\min \max_{0 \leq j \leq j^+}$" term). The square root term, when viewed as a function of $\sum_{i=1}^{j} B_i$, spans the north-east quadrant of a circumference of radius f, and so it is strictly concave decreasing in $\sum_{i=1}^{j} B_i$. Therefore, j_ℓ necessarily takes one of two values: the largest value of j such that $\phi_\ell'(\sum_{i=1}^{j} B_i) \geq 0$; or the smallest value of j such that $\phi_\ell'(\sum_{i=1}^{j} B_i) \leq 0$. More precisely, j_ℓ must equal either r or $r + 1$, where

$$\sum_{i=1}^{r} B_i \leq \frac{f}{(1 + (\ell/\kappa)^2)^{1/2}} \leq \sum_{i=1}^{r+1} B_i \tag{45}$$

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for viewed as a function of $\sum_{i=1}^{j} B_i$, $\phi_\ell(\sum_{i=1}^{j} B_i)$ is maximized when $\sum_{i=1}^{j} B_i = f/\left(1 + (\ell/\kappa)^2\right)^{1/2}$. This leads to an upper bound for the number of casualties the government can avert: the best the government can do is set $\sum_{i=1}^{j} B_i = f/\left(1 + (\ell/\kappa)^2\right)^{1/2}$ while the insurgents can inflict at most κ civilian casualties, which implies that

$$\phi_\ell(\sum_{i=1}^{j} B_i) \leq \min\left\{ f\left(\left(1 + \frac{\kappa^2}{\ell^2}\right)^{1/2} - 1\right), \kappa\right\}. \tag{46}$$

From (45), the number of engagements taken by the government decreases with ℓ and increases with κ. Large values of ℓ/κ lead to $B_1 > f(1 + (\ell/\kappa)^2)^{-1/2}$, in which case the government fights at most one battle. On the other hand, small values of ℓ/κ lead to $j_\ell = m$ (indeed from equation (45), this must happen when $\ell \leq f/(1 + (\ell/\kappa)^2)^{1/2}$). This reasoning indicates that the insurgents attempt to make ℓ as large as possible.

However, just maximizing ℓ does not tell the whole story. Indeed, looking at Equations (43) and (45), the insurgents' best option is to make the government indifferent to having $j_\ell = r$ or $j_\ell = r + 1$. This means that, for values of ℓ sufficiently large such that $j_\ell < m$, the optimal value of B_{r+1} satisfies

$$\phi_\ell\left(\sum_{i=1}^{r} B_i\right) = \phi_\ell\left(\sum_{i=1}^{r+1} B_i\right). \tag{47}$$

Moreover, on the space of feasible solutions where (45) holds, the optimal solution for the insurgency is the one that minimizes $\phi_\ell(\sum_{i=1}^{r} B_i)$ while preserving equation (47), which is analogous to making *both* ℓ and B_{r+1} large while preserving equations $(45 - 47)$.

In other words, the last battle offered by the insurgency is relatively large and

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the *net change* in casualties averted by taking versus not taking it is zero. This happens because if the government takes the $(r+1)^{st}$ town then $k_{r+1} = \kappa B_{r+1}/\ell$ extra casualties are averted, but additional soldiers' casualties are incurred because each battle is fought closer to its victory threshold. On the other hand, if the $(r+1)^{st}$ town is not taken, then the government bears extra k_{r+1} casualties, but the soldiers' casualties in the first r battles are relatively lower. Hence, in this game the government would initially be offered battles where it is clearly advantageous to attack the insurgency, and then it would be offered a relatively (to the other battles) large battle with two equally unappealing outcomes: bear a large number of civilian casualties, or take many more soldiers' casualties.

Regarding the ordering of the battle thresholds offered by the insurgency, their order does not matter when the B_i's have low variability. If the variability is large then the order of the B_i's depends on $f/(1 + \ell^2/\kappa^2)^{1/2}$. For $f/(1 + \ell^2/\kappa^2)^{1/2}$ low we have j_ℓ small, suggesting that the largest B_i's would appear first. On the other hand, when $f/(1 + \ell^2/\kappa^2)^{1/2}$ is large we have j_ℓ large too, and the insurgents will put the B_i's in increasing order, so that B_{r+1} is large. Finally, the order does not matter when $f/(1+\ell^2/\kappa^2)^{1/2}$ is so large that $j_\ell = m$.

How "close" is the knapsack game to the original problem (37–39)? Let $\mathbf{y}^*$ and $\mathbf{k}^*$ be an optimal (for the insurgency) solution to problem (37–39), which causes $\eta(\mathbf{y}^*, \mathbf{k}^*)$ casualties. Evidently, $\theta(\mathbf{y}^*, \mathbf{k}^*) \geq \eta(\mathbf{y}^*, \mathbf{k}^*)$ because the knapsack approximation is not necessarily optimal for the government (see equation (58) in the Appendix). Also, since $(\mathbf{y}^*, \mathbf{k}^*)$ are insurgent-optimal for problem (37–39), we have $\eta(\mathbf{y}^*, \mathbf{k}^*) \geq \eta(\tilde{\mathbf{y}}, \tilde{\mathbf{k}})$. Finally, $\theta(\tilde{\mathbf{y}}, \tilde{\mathbf{k}}) \geq \theta(\mathbf{y}^*, \mathbf{k}^*)$

because $(\tilde{\mathbf{y}}, \tilde{\mathbf{k}})$ is optimal for the insurgency when the government follows a knapsack approximation. Taken together, these inequalities result in

$$\theta(\tilde{\mathbf{y}}, \tilde{\mathbf{k}}) \geq \theta(\mathbf{y}^*, \mathbf{k}^*) \geq \eta(\mathbf{y}^*, \mathbf{k}^*) \geq \eta(\tilde{\mathbf{y}}, \tilde{\mathbf{k}}). \tag{48}$$

When $B_i \ll k_i$ and $\sum_i B_i > f$, Lemma 4 and its corollary (Appendix) show that the relative differences $(\theta(\tilde{\mathbf{y}}, \tilde{\mathbf{k}}) - \eta(\tilde{\mathbf{y}}, \tilde{\mathbf{k}}))/\eta(\tilde{\mathbf{y}}, \tilde{\mathbf{k}})$ and $(\theta(\mathbf{y}^*, \mathbf{k}^*) - \eta(\mathbf{y}^*, \mathbf{k}^*))/\eta(\mathbf{y}^*, \mathbf{k}^*)$ are small. But Equation (48) now implies that the relative difference $(\theta(\tilde{\mathbf{y}}, \tilde{\mathbf{k}}) - \eta(\mathbf{y}^*, \mathbf{k}^*))/\eta(\mathbf{y}^*, \mathbf{k}^*)$ is also small. This means that when the government follows the knapsack approximation, it does not do much worse than if it behaved optimally (by solving problem (15–17)), even when the insurgency always behaves optimally. This analysis suggests a robust conclusion – the government can very nearly achieve its best results possible by solving a knapsack problem, while the insurgents do best by setting k_i proportional to $B_i(y_i)$, where $\mathbf{y}$ makes both B_{r+1} and $\sum_{i=1}^m B_i$ large.

4.2 Knapsack Games in Towns and Villages

Consider again the Towns and Villages example of Section 3.3. Rather than assuming that the government can optimize its attack presuming that insurgent forces and downstream civilian casualties are distributed as in Table 1, we now model "smart" insurgents who anticipate that the government will allocate its force to maximize the objective in (27). We presume that the government has $f = 2000$ troops to allocate, while the insurgents have $\psi = 750$ fighters along with materiel (e.g. missiles, IEDs) capable of causing $\kappa = 520$ civilian casualties. The scenario and intelligence function is as in Section 3.3. We

examine the consequences of the resulting knapsack game as a function of the intelligence parameter μ.

For a given problem instance, the insurgents deploy by allocating $\tilde{y}_i$ fighters to stronghold i where the $\tilde{y}_i$'s jointly maximize $\sum_{j=1}^{m} B_j(y_j)$ ($B_j(y_j)$ is given by equation (36)), set $\tilde{k}_i = \kappa B_i(\tilde{y}_i)/\sum_{j=1}^{m} B_j(\tilde{y}_j)$, and lightly perturb the $\tilde{k}$'s to induce the government to form the knapsack ordering that minimizes casualties averted as discussed in the previous section. The government decides which strongholds to attack by solving equation (27) and allocating in proportion to those $B_i(\tilde{y}_i)$'s in the optimal battle set.

Figure 3 Here

Figure 3 presents the results. In contrast to the government's optimal ordering from Table 2 (which was Town 1, Town 2, Village 2, Village 1, and Village 3), the insurgents find it optimal to force the government to fight in the larger towns (sequence by decreasing $B_i(\tilde{y}_i)$) when intelligence is very limited ($\mu < 0.15$). For intermediate values of the intelligence parameter ($0.15 < \mu < 0.5$), the insurgents are able to reduce the number of casualties the government can avert by re-ordering the sequence according to increasing $B_i(\tilde{y}_i)$. These strategic responses result in fewer casualties averted when compared to the original Towns and Villages example with non-strategic insurgents (when $\mu \geq 0.5$, $f \geq \sum_{j=1}^{m} B_j(\tilde{y}_j)$ and the government can engage the insurgents in all strongholds). The insurgents' first-mover advantage can be considerable: when $\mu = .3$, the number of casualties the government can avert falls from 302 to 283; when $\mu = .2$, the drop is much larger – from 249 to 187 casualties averted, a 25% reduction. The

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analytical bound of equation (46) is also shown in Figure 3. This example shows how smart insurgents can readily create worst case results for the government, and not surprisingly, the damage inflicted is greater when the government's intelligence is poor.

5   Conclusions

We have investigated an important tactical question in counterinsurgency operations: how should the government optimally allocate its forces against insurgent strongholds that threaten civilian populations when both military and civilian casualties must be taken into account, and when the operations are executed with imperfect intelligence? We showed that for a given allocation of insurgents across strongholds, optimally selecting those strongholds to attack can be (approximately) accomplished with a simple knapsack rule. The forces allocated to those strongholds attacked divide the total force available in proportion to the victory thresholds B_i. These thresholds are determined in turn by the size of the insurgent force in stronghold i, the effective fire ratio β_i/α_i, and the intelligence function $p(\cdot)$. We also showed that if the insurgents anticipate government actions and have full knowledge of the battle parameters, a sequential game ensues. The solution to this game depends on the government's level of intelligence: if the government has perfect intelligence (and the insurgents know it), then the insurgents' best strategy is to concentrate its entire force and assets in one stronghold, in which case government decides whether

or not to attack depending upon the number of soldiers needed to prevent civilian casualties. With lower levels of intelligence, if the government follows a knapsack policy (which is shown to be nearly optimal), then the insurgents' optimal strategy is to spread their force and materiel in a manner that maximizes the number of soldiers the government needs to win all battles, and induce the government to select battles in a sequence that forces worst-case results. Via example, we showed that the insurgents' first-mover advantage can appreciably reduce the ability of the government to avert both civilian and battle casualties, and this problem is especially acute when the government has poor intelligence. This work thus joins other recent research in recognizing the importance of treating our adversaries as strategic players in homeland security, counterterror or counterinsurgency games ([Brown *et al* 2006], [Golany *et al* 2008], [Jacobson and Kaplan 2007], [Sandler and Arce 2007],[Zhuang and Bier 2007]).

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Appendix

This section brings theoretical support to several results stated without proof in the main body of the paper.

Error Bounds

Let

$$\varphi(V_1, \ldots, V_r) = \sum_{i=1}^{r} \left(\frac{B_i}{\sum_{j=1}^{r} B_j V_j} f - \sqrt{\left(\frac{B_i}{\sum_{j=1}^{r} B_j V_j} f \right)^2 - B_i^2} - k_i \right) V_i, \tag{49}$$

and consider the force allocation problem faced by the government:

$$\min_{V_i \in \{0,1\}} \varphi(V_1, \ldots, V_r),$$

subject to

$$\sum_{i=1}^{r} B_i V_i \leq f,$$

with optimal variables $V_1^*, \ldots, V_m^*$. Since

$$B_i V_i^* \geq \left(\frac{B_i}{\sum_{j=1}^{m} B_j V_j^*} f - \sqrt{\left(\frac{B_i}{\sum_{j=1}^{m} B_j V_j^*} f \right)^2 - B_i^2} \right) V_i^*,$$

we have

$$\varphi(V_1^*, \ldots, V_m^*) \leq \sum_{i=1}^{m} (B_i - k_i) V_i^*. \tag{50}$$

When the engagement thresholds B are small relative to the civilian casualties k, the government fights as many engagements as possible and allocates close to B forces to each engagement. This suggests considering the knapsack problem:

$$\min \sum_{i=1}^{m} (B_i - k_i) U_i \tag{51}$$

$$\text{st} \quad \sum_{i=1}^{m} B_i U_i \leq f \tag{52}$$

$$U_i \in (0, 1) \qquad \text{for } i = 1, 2, \ldots, m \tag{53}$$

Looking into (50), we must have

$$\varphi(V_1^*, \ldots, V_m^*) \le \sum_{i=1}^{m} (B_i - k_i) U_i^* \le \sum_{i=1}^{m} (B_i - k_i) V_i^*. \tag{54}$$

Suppose that $B_i \le k_i$ for all i. By the rank-ordering assumption governing the ratios k_i / B_i, the continuous relaxation of (51–53) has solution $\tilde{U}_i^* = 1$ for $i = 1, \ldots, \alpha$, where $\alpha = \max\{i : \sum_{j=1}^{i} B_j \le f\}$, and $\tilde{U}_{\alpha+1}^* = (f - \sum_{i=1}^{\alpha} B_i) / B_{\alpha+1}$. Therefore,

$$\varphi(V_1^*, \ldots, V_m^*) \le \sum_{i=1}^{\alpha} (B_i - k_i) + (B_{\alpha+1} - k_{\alpha+1}) \tilde{U}_{\alpha+1}^*, \tag{55}$$

and from (54–55) and the assumption that $B_i \le k_i$ for all i it follows that

$$\varphi(V_1^*, \ldots, V_m^*) \le \sum_{i=1}^{\alpha} (B_i - k_i) \le \sum_{i=1}^{m} (B_i - k_i) V_i^* + (k_{\alpha+1} - B_{\alpha+1}). \tag{56}$$

We call the solution determined by engaging the first α battles at level $B_1, \ldots, B_\alpha$ the *greedy* solution.

Let us now focus on problem (27), and let $W_i^* = 1$ for $i = 1, \ldots, j^*$ and $W_i^* = 0$ otherwise. (Recall that j^* is the optimal solution to problem (27).) Evidently,

$$\varphi(V_1^*, \ldots, V_r^*) \le \varphi(W_1^*, \ldots, W_r^*), \tag{57}$$

since $V_1^*, \ldots, V_r^*$ is optimal for (15–17).

Compared to a greedy solution, problem (27) always uses up the available force f, and optimizes the number of engagements (over the first α engagements). This implies that

$$\varphi(W_1^*, \ldots, W_r^*) \le \sum_{i=1}^{\alpha} (B_i - k_i).$$

This, and $(56 - 57)$ show that

$$\varphi(V_1^*,\ldots,V_r^*) \le \varphi(W_1^*,\ldots,W_r^*) \le \sum_{i=1}^{\alpha}(B_i-k_i) \le \sum_{i=1}^{m}(B_i-k_i)V_i^*+(B_{\alpha+1}-k_{\alpha+1}).$$

$$(58)$$

Hence, the relative error incurred by the greedy solution and the solution of (27) is bounded above by

$$\frac{\sum_{i=1}^{m}(B_i - k_i)V_i^* + (k_{\alpha+1} - B_{\alpha+1}) - \varphi(V_1^*,\ldots,V_r^*)}{\varphi(V_1^*,\ldots,V_r^*)},$$

which after some algebra becomes

$$\frac{\sum_{i=1}^{m} B_iV_i^* - f + \sqrt{f^2 - (\sum_{i=1}^{m} B_iV_i^*)^2} + (k_{\alpha+1} - B_{\alpha+1})}{\varphi(V_1^*,\ldots,V_r^*)}.$$

The relative error becomes smaller as $\sum_{i=1}^{m} B_iV_i^*$ approaches f, and this occurs when the engagement thresholds are small relative to the civilian casualties. This suggests considering a sequence of problems $\mathbf{P}_1,\ldots,\mathbf{P}_r$, where $\mathbf{P}_r$ comprises mr possible engagements with total force level f. The engagement thresholds and civilian casualties for $\mathbf{P}_r$ are:

$$B_{i,r} = B_1/r \text{ and } k_{i,r} = k_1 \text{ for } i = 1,\ldots,r.$$

$$\vdots$$

$$B_{i,r} = B_r/r \text{ and } k_{i,r} = k_r \text{ for } i = r(m-1)+1,\ldots,mr.$$

Then problem $\mathbf{P}_r$ is to

$$\min_{V_{i,r}\in\{0,1\}} \sum_{i=1}^{mr} \left(\frac{B_{i,r}}{\sum_{j=1}^{mr} B_{j,r}V_{j,r}}f - \sqrt{\left(\frac{B_{i,r}}{\sum_{j=1}^{mr} B_{j,r}V_{j,r}}f\right)^2 - B_{i,r}^2 - k_{i,r}} \right) V_{i,r},$$

subject to

$$\sum_{i=1}^{mr} B_{i,r}V_{i,r} \le f.$$

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Let $V_{1,r}^*, \ldots, V_{mr,r}^*$ be the optimal solution to problem $\mathbf{P}_r$. Then we have the following result.

Lemma 4 $\sum_{i=1}^{mr} B_{i,r} V_{i,r}^* \to f$ *as* $r \to \infty$.

Proof. Suppose $\sum_{i=1}^{mr} B_{i,r} V_{i,r}^* \nrightarrow f$. Then there exists $\delta \in (0, f)$ such that

$$\liminf_{r \to \infty} \sum_{i=1}^{mr} B_{i,r} V_{i,r}^* \leq \delta.$$

For r sufficiently large, there exists some $B_{i,r}$ with $V_{i,r}^* = 0$ (say $B_{\gamma,r}$) such that

$B_{\gamma,r} \leq f - \delta$. Then the force allocation

$$x_{i,r} = \left(\frac{B_{i,r}}{\sum_{j=1}^{mr} B_{j,r} V_{j,r}^*}(f - B_{\gamma,r}) - \sqrt{\left(\frac{B_{i,r}}{\sum_{j=1}^{mr} B_{j,r} V_{j,r}^*}(f - B_{\gamma,r}) \right)^2 - B_{i,r}^2} - k_{i,r} \right) V_{i,r}^*,$$

for all $i \neq \gamma$, and

$$x_{\gamma,r} = B_{\gamma,r}$$

is feasible and changes the value of the objective function in $\mathbf{P}_r$ by $\delta/r - k_{\gamma,r}$,

for some finite positive constant δ. Therefore, for r sufficiently large, $V_{i,r}^*$ cannot

be optimal, which contradicts our initial assumption. ∎

Since $\sum_{i=1}^{mr} B_{i,r} V_{i,r}^* \to f$, it is easy to see that $\liminf_{r \to \infty} \varphi(V_{1,r}^*, \ldots, V_{mr,r}^*) \geq$

f. Also, $B_{\alpha+1,r} \to 0$ as $r \to \infty$. Therefore, we have the following corollary.

Corollary 5 *The knapsack approximation relative error with respect to the optimal government response* $\varphi(V_{1,r}^*, \ldots, V_{mr,r}^*)$ *is bounded above as* $r \to \infty$:

$$\limsup_{r \to \infty} \frac{\sum_{i=1}^{mr} B_{i,r} V_{i,r}^* - f + \sqrt{f^2 - (\sum_{i=1}^{mr} B_{i,r} V_{i,r}^*)^2} + (k_{\alpha+1,r} - B_{\alpha+1,r})}{\varphi(V_{1,r}^*, \ldots, V_{mr,r}^*)} \leq \frac{\min_i \{k_i\}}{f}.$$

In general $f \gg k_i$, so that this corollary provides theoretical support to our

assertion that the greedy policy and knapsack approximation (problem (27))

yield a very small relative error.

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Proofs for Lemmas in Section 4

Proof of Lemma 1. Suppose that the $B_i(y_i)$'s are concave and that $\mathbf{a}, \mathbf{b} \in \Gamma$, and let $\gamma \in \Re$. We need to show that $\gamma \mathbf{a} + (1 - \gamma)\mathbf{b} \in \Gamma$ for all $\gamma \in [0, 1]$.

Assume, without loss of generality, that $a_1 > b_1, \ldots, a_r > b_r$, that $a_{m+1} = b_{m+1}, \ldots, a_s = b_s$, and that $a_{s+1} < b_{s+1}, \ldots, a_m < b_m$, for $1 \leq r \leq s \leq m$. By our hypotheses, the inverse functions $B_1^{-1}, \ldots, B_m^{-1}$ are well-defined and continuous. Moreover, the function $g_i(\gamma) = B_i^{-1}(\gamma a_i + (1 - \gamma)b_i)$ is increasing and convex in γ for $1 \leq i \leq r$, constant for $r + 1 \leq i \leq s$, and decreasing convex for $s + 1 \leq i \leq m$.

Therefore, the function

$$g(\gamma) = \sum_{i=1}^{m} g_i(\gamma)$$

is convex in γ. This, and the fact that $\mathbf{a}, \mathbf{b} \in \Gamma$ implies $g(1) \leq \psi$ and $g(0) \leq \psi$, results in

$$g(\gamma) \leq \max\{g(0), g(1)\} \leq \psi.$$

This means that $\gamma \mathbf{a} + (1 - \gamma)\mathbf{b}$ is the image of elements in $\mathcal{M}_\psi$ for all $0 \leq \gamma \leq 1$, and therefore $\gamma \mathbf{a} + (1 - \gamma)\mathbf{b} \in \Gamma$. ∎

Proof of Lemma 2. For $\mathbf{k} \in \mathcal{M}_\kappa$ and $\mathbf{y} \in \mathcal{M}_\psi$ arbitrary, let $\tilde{V} = \{i : k_i > 0\}$ and $V^* = \{i : V_i^* = 1\}$. If $f \geq \kappa \geq B_{i^*}(\psi)$ or $\kappa \geq f \geq B_{i^*}(\psi)$, then

$$\sum_{i \notin V^*} k_i + f - \sqrt{f^2 - \left(\sum_{i \in V^*} B_i(y_i)\right)^2} \leq f - \sqrt{f^2 - \left(\sum_{i \in \tilde{V}} B_i(y_i)\right)^2},$$

since V^* is optimal for the government. But for $y_{i^*} = \psi$ and $k_{i^*} = \kappa$, the government attacks and defeats the guerrilla (causing less casualties than by

not engaging), meaning that

$$f - \sqrt{f^2 - \left(\sum_{i \in \tilde{V}} B_i(y_i)\right)^2} \leq f - \sqrt{f^2 - B_{i^*}^2(\psi)}. \tag{59}$$

The result for (i) now follows by putting the two inequalities together.

If $f - (f^2 - B_{i^*}^2(\psi))^{1/2} \leq \kappa$ then the proof is identical to the above. If $f - (f^2 - B_{i^*}^2(\psi))^{1/2} > \kappa$ then the government does not engage the guerrilla, which causes κ civilian casualties. But when both players allocate their resources optimally, the number of casualties is bounded above by κ, and the guerrilla achieves this bound by setting $y_{i^*} = \psi$ and $k_{i^*} = \kappa$.

If $B_{i^*}(\psi) > f$, by allocating all its forces to y_{i^*} and k_{i^*} the guerrilla prevents the government from engaging it, and causes κ civilian casualties, so that their allocation is optimal. ∎

Proof of Lemma 3. Suppose $\theta(\tilde{\mathbf{y}}, \tilde{\mathbf{k}}) > 0$ and that $\tilde{k}_{j^*}/\tilde{B}_{j^*} = \tilde{k}_{j^*+1}/\tilde{B}_{j^*+1} + \epsilon$, for some $\epsilon > 0$. Then there exists $\delta > 0$ sufficiently small such that $k'_{j^*} = \tilde{k}_{j^*} - \delta$ and $k'_{j^*+1} = \tilde{k}_{j^*+1} + \delta$, with $\tilde{k}_{j^*}/\tilde{B}_{j^*} > k'_{j^*}/\tilde{B}_{j^*} > k'_{j^*+1}/\tilde{B}_{j^*+1} > \tilde{k}_{j^*+1}/\tilde{B}_{j^*+1}$. But reducing $\tilde{k}_{j^*}$ leads to

$$\sum_{i=1}^{j^*-1} k_i + k'_{j^*} - f + \left(f^2 - \left(\sum_{i=1}^{j^*} B_i(y_i)\right)^2\right)^{1/2} < \sum_{i=1}^{j^*} k_i - f + \left(f^2 - \left(\sum_{i=1}^{j^*} B_i(y_i)\right)^2\right)^{1/2} \tag{60}$$

so that casualties averted when the government engages the first j^* strongholds decrease. On the other hand, the casualties averted by fighting in the first j^*+1 strongholds does not change, since $\sum_{i=1}^{j^*-1} k_i + k'_{j^*} + k'_{j^*+1} = \sum_{i=1}^{j^*+1} k_i$. Hence,

if

$$\sum_{i=1}^{j^*} k_i - f + \left(f^2 - \left(\sum_{i=1}^{j^*} B_i(y_i) \right)^2 \right)^{1/2} > \sum_{i=1}^{j^*+1} k_i - f + \left(f^2 - \left(\sum_{i=1}^{j^*+1} B_i(y_i) \right)^2 \right)^{1/2}$$
(61)

the insurgents can improve their solution by shifting a bit of materiel (δ) from k_{j^*} to k_{j^*+1}, which is a contradiction.

It could be possible, however, that (61) holds with equality but, since now $k'_{j^*+1}/\tilde{B}_{j^*+1} > \tilde{k}_{j^*+2}/\tilde{B}_{j^*+2}$, the insurgents can shift a small amount of materiel from k'_{j^*+1} to $\tilde{k}_{j^*+2}$. Just like in Equation (60), the number of casualties averted if the government engaged the first $j^* + 1$ decreases too. In general, it could happen that $\sum_{i=1}^{j} k_i - f + (f^2 - (\sum_{i=1}^{j} B_i(y_i))^2)^{1/2}$ is constant for $j = j^*, \ldots, m$. In this case, shifting a bit of materiel from $k_{j^*} \to k_{j^*+1} \to \cdots \to k_m$ preserves the ratio orderings, and produces a net decrease in the casualties averted if the government chose to fight any of $j^*, \ldots, m-1$ battles. For the last battle, a small shift from $\tilde{y}_{m-1}$ to $\tilde{y}_m$ would preserve the ratios order and lower the casualties averted when all the m strongholds are engaged. In conclusion, $\theta(\tilde{\mathbf{y}}, \tilde{\mathbf{k}})$ can be decreased if $\tilde{k}_{j^*}/\tilde{B}_{j^*} > \tilde{k}_{j^*+1}/\tilde{B}_{j^*+1}$, so that $(\tilde{\mathbf{y}}, \tilde{\mathbf{k}})$ cannot be optimal for the insurgency.

Now we "propagate" the equal ratios property to the right. Suppose that $\tilde{k}_{j^*}/\tilde{B}_{j^*} = \tilde{k}_{j^*+1}/\tilde{B}_{j^*+1} = \tilde{k}_{j^*+2}/\tilde{B}_{j^*+2} + \epsilon$. Then we can use the same procedure as before, reducing $\tilde{k}_{j^*}$ and $\tilde{k}_{j^*+1}$, and increasing $\tilde{k}_{j^*+2}$ all the while preserving the original ratio ordering. These changes lower the original $\theta(\tilde{\mathbf{y}}, \tilde{\mathbf{k}})$, resulting in a contradiction. Hence $\tilde{k}_{j^*}/\tilde{B}_{j^*} = \tilde{k}_{j^*+1}/\tilde{B}_{j^*+1} = \tilde{k}_{j^*+2}/\tilde{B}_{j^*+2}$. This argument can be repeatedly used to show that $\tilde{k}_{j^*}/\tilde{B}_{j^*} = \cdots = \tilde{k}_m/\tilde{B}_m$.

In the last step we "propagate" the equal ratios property to the left. To wit, suppose that $\tilde{k}_{j^*-1}/\tilde{B}_{j^*-1} = \tilde{k}_{j^*}/\tilde{B}_{j^*} + \epsilon$. Then there exists $\delta > 0$ such that $k'_{j^*-1} = \tilde{k}_{j^*-1} - \delta$ and $k'_{j^*} = \tilde{k}_{j^*} + \delta$, with $\tilde{k}_{j^*-1}/\tilde{B}_{j^*-1} > k'_{j^*-1}/\tilde{B}_{j^*-1} > k'_{j^*}/\tilde{B}_{j^*} > \tilde{k}_{j^*}/\tilde{B}_{j^*}$. This change does *not* change θ, but causes $k'_{j^*}/\tilde{B}_{j^*} > \tilde{k}_{j^*+1}/\tilde{B}_{j^*+1}$. However, we already showed that θ can be lowered by reducing k'_{j^*} and increasing $\tilde{k}_{j^*+1}$. Hence, we must have $\tilde{k}_{j^*-1}/\tilde{B}_{j^*-1} = \tilde{k}_{j^*}/\tilde{B}_{j^*}$. A similar argument shows that $\tilde{k}_1/\tilde{B}_1 = \cdots = \tilde{k}_{j^*}/\tilde{B}_{j^*}$. Finally, $\sum_i \tilde{k}_i = \kappa$, results in

$$\tilde{k}_i = \kappa \frac{\tilde{B}_i}{\sum_{j=1}^m \tilde{B}_j}.$$

When $\theta(\tilde{\mathbf{y}}, \tilde{\mathbf{k}}) = 0$, carrying out the above procedure cannot increase θ, meaning that there exists a solution $(\mathbf{y}', \mathbf{k}')$ with equal ratios and for which $\theta(\mathbf{y}', \mathbf{k}') = 0$. ■

Table Legends

Table 1: Insurgency Data: Towns and Villages

Table 2: Intelligence-Dependent Optimal Force
Allocations and Casualties Averted

Table 1

Stronghold (i)	Insurgent Locations (n_i)	Insurgents (y_i)	Civilian Casualties (k_i)
Town 1	8,000	200	200
Town 2	10,000	300	200
Village 1	2,000	100	50
Village 2	1,500	100	50
Village 3	1,000	50	20

Table 2

μ	Casualties Averted	Town 1	Town 2	Village 1	Village 2	Village 3
0.00	12.14	2000.00	0.00	0.00	0.00	0.00
0.05	121.94	2000.00	0.00	0.00	0.00	0.00
0.10	155.94	2000.00	0.00	0.00	0.00	0.00
0.15	201.52	804.02	1195.98	0.00	0.00	0.00
0.20	248.80	802.93	1197.07	0.00	0.00	0.00
0.25	277.85	802.25	1197.75	0.00	0.00	0.00
0.30	302.06	671.19	1003.07	0.00	325.74	0.00
0.35	324.85	575.31	860.41	283.48	280.80	0.00
0.40	346.34	574.60	859.82	283.89	281.69	0.00
0.45	363.12	574.03	859.34	284.22	282.41	0.00
0.50	377.73	535.49	801.93	265.59	264.19	132.80
0.55	390.52	535.11	801.59	265.78	264.62	132.89
0.60	401.20	534.79	801.31	265.94	264.99	132.97
0.65	410.25	534.51	801.06	266.08	265.31	133.04
0.70	418.02	534.28	800.85	266.20	265.58	133.10
0.75	424.76	534.07	800.66	266.30	265.82	133.15
0.80	430.66	533.89	800.50	266.39	266.03	133.20
0.85	435.87	533.72	800.35	266.47	266.21	133.24
0.90	440.51	533.58	800.22	266.54	266.38	133.27
0.95	444.67	533.45	800.11	266.61	266.53	133.30
1.00	448.41	533.33	800.00	266.67	266.67	133.33

Figure Legends

Figure 1: Damage Function

Figure 2: Optimal Force Allocation and Casualties Averted

Figure 3: Casualties Averted in Knapsack Games in Towns and Villages